A strongly mimetic least-squares finite element method for the Stokes equations

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Least-Squares 101

$\mathcal{L}u = f$ in Ω

$\mathcal{R}u = h$ on Γ

$$\min_{u \in X} J(u; f, h) \equiv \frac{1}{2}\left(\|\mathcal{L}u - f\|^2_{X,\Omega} + \|\mathcal{R}u - h\|^2_{Y,\Gamma} \right)$$

$$(\mathcal{L}u, \mathcal{L}v)_\Omega + (\mathcal{R}u, \mathcal{R}v)_\Gamma = (f, \mathcal{L}u)_\Omega + (h, \mathcal{R}v)_\Gamma$$

$$\mathbf{A}\mathbf{u} = \mathbf{b}$$

Top 3 reasons people

want to do least squares:

- ☺ **Using C^0 nodal elements**
- ☺ **Avoiding inf-sup conditions**
- ☺ **Solving SPD systems**

don't want to do least squares:

- ☹ **Conservation**
- ☹ **Conservation**
- ☹ **Conservation**

We will show that

- Using **nodal elements** is not necessarily the best choice in LSFEM, and so it is arguably the **least-important advantage** attributed to least-squares methods
- By using **other** elements least-squares acquire **additional** conservation properties
- Surprisingly, this kind of least-squares turns out to be **related** to mixed methods

The Stokes system

First-order velocity-vorticity-pressure (VVP) Stokes equations

$$\begin{cases} \nabla\times\omega+\nabla p=\mathbf{f} & \text{in } \Omega \\ \nabla\times\mathbf{u}-\omega=0 & \text{in } \Omega \\ \nabla\cdot\mathbf{u}=0 & \text{in } \Omega \end{cases} \qquad \int_\Omega p\,dx=0$$

$\Omega\subset\mathbf{R}^3$ → bounded contractible domain

$\partial\Omega$ → Lipschitz continuous

Normal velocity-tangential vorticity condition

$\mathbf{n}\cdot\mathbf{u}=0$ and $\mathbf{n}\times\omega=0$ on $\partial\Omega$

Stay tuned for LSFEM with velocity BC

Function spaces and norms

$$H_0^1(\Omega)=\left\{u\in L^2(\Omega)\mid\nabla u\in\mathbf{L}^2(\Omega);\quad u=0 \text{ on } \Gamma\right\} \rightarrow \|u\|_G$$

$$H_0(\Omega,curl)=\left\{\mathbf{u}\in\mathbf{L}^2(\Omega)\mid\nabla\times\mathbf{u}\in\mathbf{L}^2(\Omega);\quad \mathbf{n}\times\mathbf{u}=0 \text{ on } \Gamma\right\} \rightarrow \|\mathbf{u}\|_C$$

$$H_0(\Omega,div)=\left\{\mathbf{u}\in\mathbf{L}^2(\Omega)\mid\nabla\cdot\mathbf{u}\in L^2(\Omega);\quad \mathbf{n}\cdot\mathbf{u}=0 \text{ on } \Gamma\right\} \rightarrow \|\mathbf{u}\|_D$$

$$H(\Omega,curl)\cap H_0(\Omega,div);\qquad H_0(\Omega,curl)\cap H(\Omega,div) \rightarrow \|\mathbf{u}\|_{CD}$$

Exact sequence property: implied by domain assumptions

$$R\mapsto H_0^1(\Omega)\xrightarrow{\nabla}H_0(\Omega,curl)\xrightarrow{\nabla\times}H_0(\Omega,div)\xrightarrow{\nabla\cdot}L_0^2(\Omega)\rightarrow 0$$

A well-posed LSFEM is a slam dunk

Stability of the VVP system

$$\|\mathbf{u}\|_{DC} + \|\omega\|_{C} + \|p\|_{G} \le C\left(\|\nabla\times\omega + \nabla p\|_0 + \|\nabla\times\mathbf{u} - \omega\|_0 + \|\nabla\cdot\mathbf{u}\|_0\right) \qquad \forall\{\mathbf{u},\omega,p\}\in X$$

$$X = H_0(\Omega,div)\cap H(\Omega,curl)\times H_0(\Omega,curl)\times H^1(\Omega)\cap L_0^2(\Omega)$$

A continuous least-squares principle (CLSP)

$$\begin{cases} J(\{\mathbf{u},\omega,p\};\mathbf{f}) = \|\nabla\times\omega + \nabla p - \mathbf{f}\|_0^2 + \|\nabla\times\mathbf{u} - \omega\|_0^2 + \|\nabla\cdot\mathbf{u}\|_0^2 \\ X = H_0(\Omega,div)\cap H(\Omega,curl)\times H_0(\Omega,curl)\times H^1(\Omega)\cap L_0^2(\Omega) \end{cases} \Rightarrow \quad \min_X J(\mathbf{u};\mathbf{f},g)$$

Stability ⇒ LS Norm-equivalence

$$J(\{\mathbf{u},\omega,p\};\mathbf{0}) \quad\propto\quad \|\mathbf{u}\|_{DC}^2 + \|\omega\|_{C}^2 + \|p\|_{G}^2 \qquad \forall\{\mathbf{u},\omega,p\}\in X$$

LS norm-equivalence ⇒ coercivity ⇒ unique least-squares solution.

✓ The LS solution coincides with the solution of the original VVP system.

✓ Any **conforming** discretization of the CLSP yields **well-posed** LSFEM

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A well-posed LSFEM is a slam dunk

Stability of the VVP system

$$\|\mathbf{u}\|_{DC}+\|\omega\|_{C}+\|p\|_{G}\le C\left(\|\nabla\times\omega+\nabla p\|_0+\|\nabla\times\mathbf{u}-\omega\|_0+\|\nabla\cdot\mathbf{u}\|_0\right)\qquad \forall\{\mathbf{u},\omega,p\}\in X$$

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A continuous least-squares principle (CLSP)

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☺ **Talk is over, let's have a beer!**

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On a second thought…maybe not

A straightforward conforming discrete least-squares principle (DLSP)

$$\begin{cases} J(\{\mathbf{u}_h,\omega_h,p_h\};\mathbf{f}) = \|\nabla\times\omega_h + \nabla p_h - \mathbf{f}\|_0^2 + \|\nabla\times\mathbf{u}_h - \omega_h\|_0^2 + \|\nabla\cdot\mathbf{u}_h\|_0^2 \\ X_h \subset H_0(\Omega,div)\cap H(\Omega,curl)\times H_0(\Omega,curl)\times H^1(\Omega)\cap L_0^2(\Omega) \end{cases}$$

The trouble with this DLSP

$$\begin{cases} \mathbf{u}_h \in H(\Omega,curl) & \Rightarrow \text{Tangential continuity} \\ \mathbf{u}_h \in H_0(\Omega,div) & \Rightarrow \text{Normal continuity} \end{cases} \Rightarrow \textbf{\textit{C}}^{\mathbf{0}}\textbf{ continuity} \Rightarrow \mathbf{u}_h \in \mathbf{H}^1(\Omega)\cap H_0(\Omega,div)$$

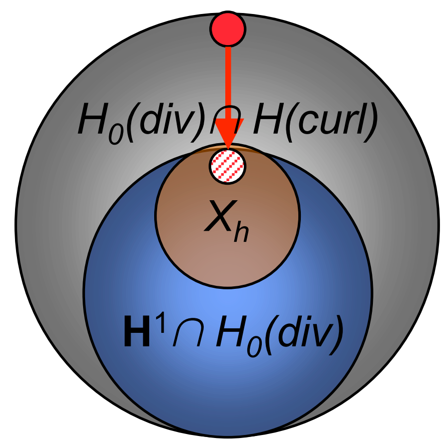
Why is this bad?

Costabel (1991) shows that unless Ω has smooth boundary or is a convex polyhedron, $\mathbf{H}^1\cap H_0(div)$ may have

infinite co-dimension in $H_0(div)\cap H(curl)$

⇒ a C^0 (nodal) finite element space may lose approximability property in $H_0(div)\cap H(curl)$, i.e., **solution will not converge.**

⇒ Mixed methods can solve this, but we have another approach…



Back to the drawing board

What to do about the velocity: Give up on ***u*** being curl-conforming

$$
\begin{cases}
J_h\left(\{\mathbf{u}_h,\omega_h,p_h\};\mathbf{f}\right) = \|\nabla\times\omega_h + \nabla p_h - \mathbf{f}\|_0^2 + \|\nabla_h^*\times\mathbf{u}_h - \omega_h\|_0^2 + \|\nabla\cdot\mathbf{u}_h\|_0^2 \\
X_h \subset H_0(\Omega,div)\cap \cancel{H(\Omega,curl)} \times H_0(\Omega,curl)\times H^1(\Omega)\cap L_0^2(\Omega)
\end{cases}
$$

We gain some and loose some:

- div-conforming **velocity**: natural for the normal velocity boundary condition
- div-conforming **velocity** : not in the domain of curl - need discrete approximation!

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The LSFEM is already "semi-conforming" so why stop here?

$$
\begin{cases}
J_h\left(\{\mathbf{u}_h,\omega_h,p_h\};\mathbf{f}\right) = \|\nabla\times\omega_h + \nabla_h^* p_h - \mathbf{f}\|_0^2 + \|\nabla_h^*\times\mathbf{u}_h - \omega_h\|_0^2 + \|\nabla\cdot\mathbf{u}_h\|_0^2 \\
X_h \subset H_0(\Omega,div)\times H_0(\Omega,curl)\times \cancel{H^1(\Omega)}\cap L_0^2(\Omega)
\end{cases}
$$

We gain some and loose some:

- discontinuous **pressure**: allows us to define "strongly compatible" method
- discontinuous **pressure**: not in the domain of grad - need discrete approximation!

Construction of semi-conforming LSFEM

Ingredient 1: a finite element De Rham complex

$$
\begin{array}{ccccccc}
H_0^1(\Omega) & \xrightarrow{\nabla} & H_0(\Omega,curl) & \xrightarrow{\nabla\times} & H_0(\Omega,div) & \xrightarrow{\nabla\cdot} & L_0^2(\Omega) \\
\Pi_G \downarrow & & \Pi_C \downarrow & & \Pi_D \downarrow & & \Pi_S \downarrow \\
G_0^h(\Omega) & \xrightarrow{\nabla} & \mathbf{C}_0^h(\Omega) & \xrightarrow{\nabla\times} & \mathbf{D}_0^h(\Omega) & \xrightarrow{\nabla\cdot} & S_0^h(\Omega)
\end{array}
$$

$\Pi_G,\quad \Pi_C,\quad \Pi_D,\quad \Pi_S,$

Bounded projection operators with Commuting Diagram Property

Ingredient 2: discrete curl and grad operators

$$\nabla_h^* \times \; \mathbf{D}_0^h(\Omega) \to \mathbf{C}_0^h(\Omega) \quad \mathbf{u}_h = \nabla_h^* \times \mathbf{w}_h \Leftrightarrow (\mathbf{u}_h, \mathbf{v}_h)_0 = (\mathbf{w}_h, \nabla \times \mathbf{v}_h)_0 \quad \forall \mathbf{v}_h \in \mathbf{C}_0^h(\Omega)$$

$$\nabla_h^* \quad S_0^h(\Omega) \to \mathbf{D}_0^h(\Omega) \quad \mathbf{u}_h = \nabla_h^* p_h \Leftrightarrow (\mathbf{u}_h, \mathbf{v}_h)_0 = -(p_h, \nabla \cdot \mathbf{v}_h)_0 \quad \forall \mathbf{v}_h \in \mathbf{D}_0^h(\Omega)$$

Valuable property (Discrete Friedrichs Inequality)

$$\|\mathbf{u}_h\|_{DC^*} \le C\left(\|\nabla_h^* \times \mathbf{u}_h\|_0 + \|\nabla \cdot \mathbf{u}_h\|_0\right) \qquad \forall \mathbf{u}_h \in \mathbf{D}_0^h$$

(effect of compatibility)

where $\|\mathbf{u}_h\|_{DC^*}^2 = \|\mathbf{u}_h\|_0^2 + \|\nabla_h^* \times \mathbf{u}_h\|_0^2 + \|\nabla \cdot \mathbf{u}_h\|_0^2 \qquad \forall \mathbf{u}_h \in \mathbf{D}_0^h$

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Is the semi-conforming LSFEM any good?

Theorem (discrete stability) $\forall\{\mathbf{u}_h,\omega_h,p_h\}\in X_h=\mathbf{D}_0^h(\Omega)\times\mathbf{C}_0^h(\Omega)\times S_0^h(\Omega)$

$$\|\mathbf{u}_h\|_{DC^*}+\|\omega_h\|_C+\|p_h\|_{G^*}\le C\left(\|\nabla\times\omega_h+\nabla_h^* p_h\|_0+\|\nabla_h^*\times\mathbf{u}_h-\omega_h\|_0+\|\nabla\cdot\mathbf{u}_h\|_0\right)$$

Proof. Using compatible FE allows to repeat the proof from the continuous case!

Discrete Stability ⇒ Discrete Norm-equivalence

$$J_h(\{\mathbf{u}_h,\omega_h,p_h\};\mathbf{0}) \quad\propto\quad \|\mathbf{u}_h\|_{DC^*}^2+\|\omega_h\|_C^2+\|p\|_{G^*}^2 \qquad \forall\{\mathbf{u}_h,\omega_h,p_h\}\in X_h$$

LS norm-equivalence ⇒ coercivity ⇒ unique least-squares solution.

✓ The LS solution coincides with the solution of a **mimetic** VVP system:

$$\begin{cases}\nabla\times\omega_h+\nabla_h^* p=\pi_D\mathbf{f} & \text{in }\Omega\\ \nabla_h^*\cdot\omega_h=0 & \text{in }\Omega\\ \nabla_h^*\times\mathbf{u}_h-\omega_h=0 & \text{in }\Omega\\ \nabla\cdot\mathbf{u}_h=0 & \text{in }\Omega\end{cases}$$

← “exact” momentum equation

← “redundant” equation implied by vorticity def..

← vorticity is discrete curl of velocity

← divergence free velocity!

This is why we call the method **strongly mimetic!**

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What about the accuracy?

Theorem

The least-squares solution $\{\mathbf{u}_h, \omega_h, p_h\} \in X_h = \mathbf{D}_0^h(\Omega) \times \mathbf{C}_0^h(\Omega) \times S_0^h(\Omega)$ satisfies the error estimates

$$\|\nabla \times (\omega - \omega_h)\|_0 \le \inf_{\zeta_h \in \mathbf{C}_0^h} \|\nabla \times (\omega - \zeta_h)\|_0$$

$$\|\nabla \times \mathbf{u} - \nabla_h^* \times \mathbf{u}_h\|_0 \le 2\|\nabla \times \mathbf{u} - \pi_C \nabla \times \mathbf{u}\|_0 + \|\omega - \omega_h\|_0$$

$$\|\nabla p - \nabla_h^* p_h\|_0 \le 2\|\nabla p - \pi_D \nabla p\|_0 + \|\nabla \times (\omega - \omega_h)\|_0$$

If $\mathbf{C}_0^h$ is Nedelec space of the **2nd kind** and Ω is convex polyhedron.

$$\|\omega - \omega_h\|_0 \le \begin{cases} C(h + h^{1-\alpha})\|\nabla \times (\omega - \omega_h)\|_0 + \inf_{\zeta_h \in \mathbf{C}_0^h} \left\{ \|\omega - \zeta_h\|_0 + h^{1-\alpha}\|\nabla \times (\omega - \zeta_h)\|_0 \right\} \\ C(1 + h)\|\nabla \times (\omega - \omega_h)\|_0 + \inf_{\zeta_h \in \mathbf{C}_0^h} \left\{ \|\omega - \zeta_h\|_0 + \|\nabla \times (\omega - \zeta_h)\|_0 \right\} \end{cases}$$

$\alpha = \frac{3}{2} - \frac{3}{s} \quad s > 2$

depends on Ω

If $\mathbf{C}_0^h$ is Nedelec space of the **1st kind**

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \le \|\mathbf{u} - \pi_D \mathbf{u}\|_0 + \|\Pi_D \mathbf{u} - \pi_D \mathbf{u}\|_0 + \|\omega - \omega_h\|_0$$

Rates of convergence

Order of the velocity depends on the kind of the vorticity space:

If $\mathbf{C}_0^h$ is Nedelec space of the **1st kind**

$$X_h = \mathbf{D}_0^{r-1} \times \mathbf{C}_0^{r-} \times S_0^{r-2} \quad \text{or} \quad \mathbf{D}_0^{r-} \times \mathbf{C}_0^{r-} \times S_0^{r-1}$$

$$\|\nabla \times (\omega - \omega_h)\|_0 \le C h^r \|\nabla \times \omega\|_r$$

$$\|\omega - \omega_h\|_0 \le C h^r \left(\|\omega\|_r + \|\nabla \times \omega\|_r\right)$$

$$\|\nabla \times \mathbf{u} - \nabla_h^* \times \mathbf{u}_h\|_0 \le C h^r \left(\|\nabla \times \mathbf{u}\|_r + \|\omega\|_r + \|\nabla \times \omega\|_r\right)$$

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \le C h^r \left(\|\mathbf{u}\|_r + \|\omega\|_r + \|\nabla \times \omega\|_r\right)$$

Pressure is independent of the kind of the vorticity space:

$$\|\nabla p - \nabla_h^* p_h\|_0 \le C h^r \left(\|\nabla p\|_r + \|\nabla \times \omega\|_r\right)$$

If $\mathbf{C}_0^h$ is Nedelec space of the **2nd kind**

$$X_h = \mathbf{D}_0^{r-1} \times \mathbf{C}_0^{r} \times S_0^{r-2} \quad \text{or} \quad \mathbf{D}_0^{r-} \times \mathbf{C}_0^{r} \times S_0^{r-1}$$

$$\|\nabla \times (\omega - \omega_h)\|_0 \le C h^r \|\nabla \times \omega\|_r$$

$$\|\omega - \omega_h\|_0 \le C h^{r+1} \left(\|\omega\|_{r+1} + h^{-\alpha} \|\nabla \times \omega\|_r\right)$$

$$\|\nabla \times \mathbf{u} - \nabla_h^* \times \mathbf{u}_h\|_0 \le C h^{r+1} \left(\|\nabla \times \mathbf{u}\|_{r+1} + \|\omega\|_{r+1} + h^{-\alpha} \|\nabla \times \omega\|_r\right)$$

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \le C h^r \left(\|\mathbf{u}\|_r + h \|\omega\|_{r+1} + h^{1-\alpha} \|\nabla \times \omega\|_r\right)$$

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Connection with a mixed method

Theorem

Consider the following mixed **vorticity-velocity potential-pressure** formulation:

Seek $\{\xi_h, \omega_h, p_h\} \in X_{MIX} = \mathbf{C}_0^h(\Omega) \cap \mathrm{N}(\nabla\times)^\perp \times \mathbf{C}_0^h(\Omega) \times S_0^h(\Omega)$ such that

$$\begin{cases} (\nabla\times\omega_h, \nabla\times\tilde{\omega}_h) = (\mathbf{f}, \nabla\times\tilde{\omega}_h) & \forall\tilde{\omega}_h \in \mathbf{C}_0^h(\Omega) \\ (\nabla\times\xi_h, \nabla\times\tilde{\xi}_h) = (\omega_h, \tilde{\xi}_h) & \forall\tilde{\xi}_h \in \mathbf{C}_0^h(\Omega) \\ (\nabla_h^* p_h, \nabla_h^* \tilde{p}_h) = (\mathbf{f}, \nabla_h^* \tilde{p}_h) & \forall\tilde{p}_h \in S_0^h(\Omega) \end{cases}$$

If $\mathbf{D}_0^h(\Omega)$ is a div-compatible FE space that contains the range of curl acting on $\mathbf{C}_0^h(\Omega)$, then

$$\{\mathbf{u}_h, \omega_h, p_h\} = \{\nabla\times\xi_h, \omega_h, p_h\} \in X_{LS} = \mathbf{D}_0^h(\Omega) \times \mathbf{C}_0^h(\Omega) \times S_0^h(\Omega)$$

is solution of the strongly mimetic LSFEM.

- Similar (up to pressure space) to a mixed method by Girault (*Math. Comp.51* 1988)
- Requires basis for the orthogonal complement of the nullspace $\mathrm{N}(\nabla\times)^\perp$
- Characterization of $\mathrm{N}(\nabla\times)^\perp$ not as straightforward as that for $\mathrm{N}(\nabla\times) = \nabla G_0^h(\Omega)$
- Strongly mimetic LSFEM is easier to implement

What about solving the equations?

The pressure equation can be solved independently

$$\left(\nabla_h^* p_h, \nabla_h^* \tilde{p}_h\right) = \left(\mathbf{f}, \nabla_h^* \tilde{p}_h\right) \qquad \forall \tilde{p}_h \in S_0^h(\Omega)$$

Vorticity and velocity can be computed from the following weak problems

$$\left(\nabla \times \omega_h, \nabla \times \xi_h\right) + \left(\nabla_h^* \cdot \omega_h, \nabla_h^* \cdot \xi_h\right) = \left(\mathbf{f}, \nabla \times \xi_h\right) \quad \forall \xi_h \in \mathbf{C}_0^h(\Omega)$$

$$\left(\nabla_h^* \times \mathbf{u}_h, \nabla_h^* \times \mathbf{v}_h\right) + \left(\nabla \cdot \mathbf{u}_h, \nabla \cdot \mathbf{v}_h\right) = \left(\omega_h, \nabla_h^* \times \mathbf{v}_h\right) \quad \forall \mathbf{v}_h \in \mathbf{D}_0^h(\Omega)$$

which are the Euler-Lagrange equations of a

curl-conforming and div-conforming

$$\begin{cases} J(\omega_h; \mathbf{f}) = \|\nabla \times \omega_h - \mathbf{f}\|_0^2 + \|\nabla_h^* \cdot \omega_h\|_{0,\Theta_0}^2 \\ X_h = \mathbf{C}_0^h(\Omega) \end{cases}$$

$$\begin{cases} J(\mathbf{u}_h; \omega_h) = \|\nabla_h^* \times \mathbf{u}_h - \omega_h\|_0^2 + \|\nabla \cdot \mathbf{u}_h\|_0^2 \\ X_h = \mathbf{D}_0^h(\Omega) \end{cases}$$

LSFEMS for two complementary div-curl systems.

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We now have efficient AMG for these problems!

For clarity we explain the lowest-order case for the vorticity system

$\mathbf{C}_0^h(\Omega) \rightarrow$ the lowest-order Nedelec space of the 1st kind

$G_0^h(\Omega) \rightarrow$ the lowest-order nodal C^0 space

Discrete least-squares problem

$$\left(\nabla\times\omega_h, \nabla\times\xi_h\right)_0 + \left(\nabla_h^*\cdot\omega_h, \nabla_h^*\cdot\xi_h\right)_0 = \left(\mathbf{f}, \nabla\times\xi_h\right)_0 \quad \forall\xi_h \in \mathbf{C}_0^h$$

$$\mathbf{K} + \mathbf{M}_C\mathbf{D}_{VE}\mathbf{M}_G^{-1}\mathbf{D}_{VE}^T\mathbf{M}_C = \mathbf{f}$$

curl-curl + grad-div matrices

where

$\mathbf{M}_C$ **curl-conforming mass matrix**

$\mathbf{M}_G$ **grad-conforming mass matrix**

$\mathbf{D}_{VE}$ **vertex-to-edge incidence matrix**

Note: Can use any *O(h)* approximation for $\mathbf{M}_G$
We will use mass lumping.

AMG solver for this system is available as a subsolver of the eddy current AMG in Bochev, Hu, Tuminaro and Siefert, SISC 2008.

Conclusions

Even in least-squares:

Compatibility pays and there's no free lunch

☺ **Compatible FE allow to formulate LSFEMs for the Stokes equations with**

- – Divergence-free velocity
- – Discrete momentum equation
- – Proper relationships between the variables ("redundant" equation)
- – Robust even for rough solutions

☹ **Forming and solving the resulting linear systems requires advanced tools:**

- – Formally linear systems include inverse mass matrix
- – The div-curl systems require sophisticated AMG solver

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But we can have a beer now!

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